Spacetime Dependent Lagrangians and Dyon Charge Quantisation in Abelian p-Form Theories

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Abstract

The spacetime dependent lagrangian formalism of references [1-2] is used to obtain the Deser-Gomberoff-Henneaux-Teitelboim results [3] for dyon charge quantisation for abelian p-form theories in dimensions $D = 2(p+1)$ for both even and odd p.

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In [1] the formalism of spacetime dependent lagrangians was used to obtain the Dirac quantisation condition. Here we shall follow the same method to obtain the results of Deser,Gomberoff,Henneaux and Teitelboim [3] regarding dyon charge quantisation in abelian , $p-$form theories. Our results will be obtained from a simple generalisation of the lagrangian constructed in [1] and a generalisation of the interaction terms using the completely antisymmetric symmetric tensor $\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the symmetric tensor $\rho_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We first briefly review the relevant material of [1].

Let the lagrangian L' be a function of fields η_ρ, their derivatives $\eta_{\rho,\nu}$ *and the spacetime coordinates* x_ν, i.e. $L' = L'(\eta_\rho, \eta_{\rho,\nu}, x_\nu)$. Variational principle yields :

$$\int dV \left(\partial_\eta L' - \partial_\mu \partial_{\partial_\mu \eta} L' \right) = 0 \tag{1}$$

Assuming a separation of variables : $L'(\eta_\sigma, \eta_{\sigma,\nu}, ..x_\nu) = L(\eta_\sigma, \eta_{\sigma,\nu})\Lambda(x_\nu)$ ($\Lambda(x_\nu)$ is the x_ν dependent part and is a finite non-vanishing function) gives

$$\int dV \left(\partial_\eta (L\Lambda) - \partial_\mu \partial_{\partial_\mu \eta} (L\Lambda) \right) = 0 \tag{2}$$

We will be confined to classical solutions of theories where the fields do not couple to gravity. Then Λ is not dynamical and is a finite, non-vanishing function of x_ν multiplying the primitive lagrangian L. It is like an external field and equations of motion for Λ meaningless. Duality invariance is related to finiteness of Λ. When equations of motion are duality invariant, finiteness of Λ on the spatial boundary at infinity leads to new solutions for the fields. Poincare invariance and duality invariance is achieved through same behaviour of Λ. The finite behaviour of Λ on the boundary *encodes the exotic solutions of the theory within the boundary.* In this way we are reminded of the holographic principle [5].

In[1] we considered a $U(1) \otimes U(1)$ gauge invariant theory.A_μ and B_μ were four-vector potentials corresponding to electric (e) and magnetic (g) charges; $F_{\mu\nu}$, $G_{\mu\nu}$ were the respective field strengths; j_μ, k_μ were the electric and magnetic (current) sources with interactions between respective currents and potentials introduced in the usual way

$$L_1 = -(1/4)F^{\mu\nu}F_{\mu\nu} - (1/4)G^{\mu\nu}G_{\mu\nu} - j^\mu A_\mu - k^\mu B_\mu \tag{3a}$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$; $G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$; $\tilde{G}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$; $\partial^\mu j_\mu = \partial^\mu k_\mu = 0$ (current conservation) ; $\partial^\mu A_\mu = \partial^\mu B_\mu = 0$ (transversality) ; $\partial^\mu F_{\mu\nu} = j_\nu$; $\partial^\mu \tilde{F}_{\mu\nu} = 0$; $\partial^\mu G_{\mu\nu} = k_\nu$; $\partial^\mu \tilde{G}_{\mu\nu} = 0$. Defining (note that $\tilde{\tilde{F}} = -F$ and $\tilde{\tilde{G}} = -G$)

$$\xi^{\mu\nu} = F^{\mu\nu} + \tilde{G}^{\mu\nu} \ ; \tilde{\xi}^{\mu\nu} = F^{\mu\nu} - \tilde{G}^{\mu\nu} \tag{3b}$$

means

$$\partial^\mu \xi_{\mu\nu} = j_\nu \ ; \partial^\mu \tilde{\xi}_{\mu\nu} = -k_\nu \tag{3c}$$

A complex interaction term $if(\Lambda)\alpha A^\mu B_\mu j^\nu k_\nu$ was introduced where $f(\Lambda)$ was a dimensionless function of Λ, and the spacetime dependent lagrangian was written as

$$L = [-(1/4)F^{\mu\nu}F_{\mu\nu} - (1/4)G^{\mu\nu}G_{\mu\nu} - j^\mu A_\mu - k^\mu A_\mu + if(\Lambda)\alpha A^\mu B_\mu j^\nu k_\nu]\Lambda(x) \tag{3c}$$

Equations of motion using (2) were set up, duality invariance imposed and the solution for Λ obtained for appropriate sources j_μ, k_μ. Finiteness of Λ at $r \to \infty$ led to the Dirac quantisation condition. The $U(1) \otimes U(1)$ invariance of the original theory was broken.

We now use the above procedure to obtain the dyon charge quantisation condition for abelian p-form theories. First consider dimension $D = 4$. Then

$p = 1$. There are now two objects, each of which carries both electric (e) and magnetic (g) charges.Accordingly, there will be two F 's, two G 's two A 's, two B 's and two j 's and two k 's. Let the index $a = 1,2$ denote this. We next choose the interaction term as $if(\Lambda)\epsilon^{bc}\alpha A^{\mu}_{a} B_{a\ \mu} j^{\nu}_{b} k_{c\ \nu}$. Then the generalisation of the lagrangian ($3a$) becomes

$$L = [-(1/4)F^{\mu\nu}_{a} F_{a\ \mu\nu} - (1/4)G^{\mu\nu}_{a} G_{a\ \mu\nu} - j^{\mu}_{a} A_{a\ \mu} - k^{\mu}_{a} B_{a\ \mu}$$

$$+i\ f(\Lambda)\ \alpha\ \epsilon^{bc}\ A^{\mu}_{a}\ B_{a\ \mu}\ j^{\nu}_{b}\ k_{c\ \nu}]\ \Lambda(x) \tag{4}$$

As before $\xi^{\mu\nu}_{a} = F^{\mu\nu}_{a} + \tilde{G}^{\mu\nu}_{a}\ ; \tilde{\xi}^{\mu\nu}_{a} = F^{\mu\nu}_{a} - \tilde{G}^{\mu\nu}_{a}$. Equations of motion that follow from (2) are (for each $a = 1,2$):

$$\Lambda(\partial^{\mu}\xi_{a\ \mu\nu}) + [(\partial^{\mu}\Lambda)F_{a\ \mu\nu} - \Lambda(j_{a\ \nu} + ic_{a\ \nu})] = 0 \tag{5a}$$

$$\Lambda(\partial^{\mu}\tilde{\xi}_{a\ \mu\nu}) - [(\partial^{\mu}\Lambda)G_{a\ \mu\nu} - \Lambda(k_{a\ \nu} + id_{a\ \nu})] = 0 \tag{5b}$$

where $c_{a\ \nu} = f(\Lambda)\alpha\epsilon^{bc} j^{\mu}_{b} k_{c\mu} B_{a\ \nu}\ ; d_{a\ \nu} = f(\Lambda)\alpha\epsilon^{bc} j^{\mu}_{b} k_{c\mu} A_{a\ \nu}$. Duality invariance means $\partial^{\mu}\xi_{a\ \mu\nu} = 0$ and $\partial^{\mu}\tilde{\xi}_{a\ \mu\nu} = 0$. This therefore implies

$$(\partial^{\mu}\Lambda)F_{a\ \mu\nu} - \Lambda(j_{a\ \nu} + ic_{a\ \nu}) = 0 \tag{6a}$$

$$(\partial^{\mu}\Lambda)G_{a\ \mu\nu} - \Lambda(k_{a\ \nu} + id_{a\ \nu}) = 0 \tag{6b}$$

To solve the above for specific sources we take $j^{\nu}_{a} = e_a \int dx^{\nu}\delta^4(x)\ ; k^{\nu}_{a} = g_a \int dx^{\nu}\delta(x_3 - b)\delta^3(x)$. Now assume $\Lambda = \Lambda(x_3)$ and that only the $\nu = 0$ component of the sources are present so that $j^0_a = e_a\delta(x_1)\delta(x_2)\delta(x_3)\ ; k^0_a = g_a\delta(x_1)\delta(x_2)\delta(x_3 - b)$. Then we get for $\nu = 0,1,2$

$$(\partial^3\Lambda)F_{a\ 3\nu} = \Lambda(j_{a\ \nu} + ic_{a\ \nu}) \tag{7a}$$

$$(\partial^3\Lambda)G_{a\ 3\nu} = \Lambda(k_{a\ \nu} + id_{a\ \nu}) \tag{7b}$$

For $\nu = 3, F_{a \ 33} = G_{a \ 33} = 0$ for all a, and the solutions to $(7a)$ and $(7b)$ for $\nu = 0$ are:

$$\begin{aligned}
\Lambda_\infty \ &= \Lambda_{-\infty} exp[e_a \delta(x_1)\delta(x_2)/F_{a \ 30}(x_0, x_1, x_2, 0)] \\
&\quad exp[if(\Lambda)\alpha\epsilon_{bc}e_b g_c P_{a \ 0}(x_0, x_1, x_2, b)] \\
&= \Lambda_{-\infty} exp[e_a \delta(x_1)\delta(x_2)/F_{a \ 30}(x_0, x_1, x_2, 0)] \\
&\quad exp[if(\Lambda)\alpha(e_1 g_2 - e_2 g_1)P_{a \ 0}(x_0, x_1, x_2, b)]
\end{aligned} \tag{8a}$$

$$\begin{aligned}
\Lambda_\infty \ &= \Lambda_{-\infty} exp[g_a \delta(x_1)\delta(x_2)/G_{a \ 30}(x_0, x_1, x_2, 0)] \\
&\quad exp[if(\Lambda)\alpha\epsilon_{bc}e_b g_c Q_{a \ 0}(x_0, x_1, x_2, b)] \\
&= \Lambda_{-\infty} exp[e_a \delta(x_1)\delta(x_2)/G_{a \ 30}(x_0, x_1, x_2, 0)] \\
&\quad exp[if(\Lambda)\alpha(e_1 g_2 - e_2 g_1)Q_{a \ 0}(x_0, x_1, x_2, b)]
\end{aligned} \tag{8b}$$

$$P_{a \ 0}(x_0, x_1, x_2, b) = (\delta(x_1))^2)(\delta(x_2))^2\delta(b)B_{a0}(x_0, x_1, x_2, b)/F_{a \ 30}(x_0, x_1, x_2, b) \tag{9a}$$

$$Q_{a \ 0}(x_0, x_1, x_2, b) = (\delta(x_1))^2)(\delta(x_2))^2\delta(b)A_{a0}(x_0, x_1, x_2, b)/G_{a \ 30}(x_0, x_1, x_2, b) \tag{9b}$$

Proceeding as in ref.[1], choose $\Lambda_\infty = \Lambda_{-\infty} = 1$ and consider the set of equations $(8a)$ and $(9a)$. The two exponentials must reduce to unity. For the first exponential this implies the Dirac string configuration where $F_{a \ 30} \to \infty$, and so the exponential becomes unity. For the second exponential, the numerator in $(9a)$ has singular δ-functions and together with $B_{a \ 30} \to \infty$ since $F_{a \ 30} \to \infty$.So second exponential is unity if $exp[if(\Lambda)\alpha(e_1 g_2 - e_2 g_1)P_{a \ 0}] = 1$,i.e. $exp[if(\Lambda)\alpha(e_1 g_2 - e_2 g_1)]^{P_{a \ 0}} = 1$ (as $P_{a \ 0}$ is finite). Therefore

$$f(\Lambda)\alpha(e_1 g_2 - e_2 g_1) = 2\pi n \tag{10}$$

All the above results are true in each sector, *viz.*, $a = 1, 2$. As in ref. [1], there are two possibilities: (a)$f(\Lambda) = 0$. Then the $U(1) \otimes U(1)$ invariance in each sector of L is unbroken and we have the Dirac string configuration from the first exponential $F_{a\ 30} \to \infty$. (b)$f(\Lambda) = a\ finite\ constant$. Then the $U(1) \otimes U(1)$ invariance in each sector of L is broken and putting $\alpha = (\hbar)^{-1}$, *we get the Dirac quantisation condition for dyons.* For $\nu = 1, 2$ a similar analysis will again lead to (10) and similarly for the set of equations $8(b)$ and $(9b)$. Note that we have taken the same function Λ in each sector of the theory. This is justifiable from the fact that finally we proceed to the case of Λ becoming unity in each sector.

Now consider dimension $D = 6$. This means $p = 2$. So we have 2-form potentials $A_a^{\mu\nu}\ ;\ B_a^{\mu\nu}$. Then each of the antisymmetric field strengths F, G will be a 3-form in the Lorentz indices and we have the following constructions:

$$3-form\ \ field\ \ stengths:\ \ F_a^{\mu\nu\sigma} = \partial^\mu A_a^{\nu\sigma} - \partial^\nu A_a^{\mu\sigma} + \partial^\sigma A_a^{\mu\nu}$$

$$G_a^{\mu\nu\sigma} = \partial^\mu B_a^{\nu\sigma} - \partial^\nu B_a^{\mu\sigma} + \partial^\sigma B_a^{\mu\nu}$$

$$\xi_a^{\mu\nu\sigma} = F_a^{\mu\nu\sigma} + \tilde{G}_a^{\mu\nu\sigma}\ \ ;\ \ \tilde{\xi}_a^{\mu\nu\sigma} = F_a^{\mu\nu\sigma} - \tilde{G}_a^{\mu\nu\sigma}$$

$$2-form\ \ antisymmetric\ \ potentials\ \ A_a^{\mu\nu}\ \ ;\ \ B_a^{\mu\nu}\ \ (antisymmmetric\ \ w.r.t.\ \ \mu,\nu)$$

$$Dual:\ \ \tilde{F}_a^{\mu\nu\sigma} = (1/3!)\epsilon^{\mu\nu\sigma\alpha\beta\gamma}F_{a\ \ \alpha\beta\gamma}\ \ ;\ \ \tilde{G}_a^{\mu\nu\sigma} = (1/3!)\epsilon^{\mu\nu\sigma\alpha\beta\gamma}G_{a\ \ \alpha\beta\gamma}$$

$$2-form\ \ currents:\ \ j_a^{\mu\nu} = e_a \int dx^\mu \wedge dx^\nu \delta^6(x)\ \ ;\ \ k_a^{\mu\nu} = g_a \int dx^\mu \wedge dx^\nu \delta^6(x)$$

Now assume that the only non-zero currents are $j_a^{0\nu}$ and $k_a^{0\nu}$; $\Lambda = \Lambda(x_5)$ and take the lagrangian as

$$L = [-(1/12)F_a^{\mu\nu}F_{a\ \ \mu\nu} - (1/12)G_a^{\mu\nu}G_{a\ \ \mu\nu} - (1/4)j_a^\mu A_{a\ \ \mu} - (1/4)k_a^\mu B_{a\ \ \mu}$$
$$+i\ \ f(\Lambda)\ \ \alpha\ \ \rho^{bc}\ \ A_a^\mu\ \ B_{a\ \ \mu}\ \ j_b^\nu\ \ k_{c\ \ \nu}]\ \ \Lambda(x) \qquad (11)$$

where the matrix $\rho_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is then straigtforward to obtain the dyon quantisation condition by proceeding exactly as before and the result is

$$e_1 g_2 + e_2 g_1 = 2\pi n\hbar \tag{12}$$

Thus the quantisation condition depends on whether p is odd or even.

In fact,the above procedure can be generalised to arbitrary p-form fields by constructing appropriate field strengths F, G and choosing the lagrangian as

$$\begin{aligned} L = [&-(1/2)(1/(p+1)!)F_a^{\mu_1..\mu_{p+1}} F_{a\ \ \mu_1..\mu_{p+1}} - (1/2)(1/(p+1)!)G_a^{\mu\nu} G_{a\ \ \mu\nu} \\ &-(1/2)(1/p!)j_a^{\mu_1..\mu_p} A_{a\ \ \mu_1..\mu_p} - (1/2)(1/p!)k_a^{\mu_1..\mu_p} B_{a\ \ \mu_1..\mu_p} \\ &+i\ \ f(\Lambda)\ \ \alpha\ \ \Omega^{bc}\ \ A_a^{\mu_1..\mu_p}\ \ B_{a\ \ \mu_1..\mu_p}\ \ j_b^{\mu_1..\mu_p}\ \ k_{c\ \ \mu_1..\mu_p}]\ \ \Lambda(x) \end{aligned} \tag{13}$$

where the matrix

$$\Omega_{ab} = (1/2)[(1+(-1)^{p+1})\epsilon_{ab} \ \ + \ \ (1+(-1)^p)\rho_{ab}] \tag{14}$$

Currents will be defined as

$$j_a^{\mu_1..\mu_p} = e_a \int dx^{\mu_1} \wedge dx^{\mu_2} .... \wedge dx^{\mu_p} \tag{15a}$$

$$k_a^{\mu_1..\mu_p} = g_a \int dx^{\mu_1} \wedge dx^{\mu_2} .... \wedge dx^{\mu_p} \tag{15b}$$

Assuming as before that the only non vanishing currents are $j_a^{0\mu_1..\mu_{p-1}}$ and $k_a^{0\mu_1..\mu_{p-1}}$ and $\Lambda = \Lambda(x_i)$,where i is some spatial coordinate, one can solve the relevant equations to get the dyon quantisation condition again. Depending on whether p is odd or even we will have

$$e_1 g_2 + (-1)^p e_2 g_1 = 2\pi n\hbar \tag{16}$$

We mention that for odd p we will have anti-selfdual field strengths, while for even p we will have selfdual field strengths.

In conclusion, we have shown that the spacetime dependent lagrangian formulation of electromagnetic duality can also accommodate the results of [3]. The dependence of the quantisation condition on p [3,4] is also accommodated. In our scheme this has to do with the fact that coupling in the interaction lagrangian depends on p through the matrix Ω_{ab}. The importance of the dyon charge quantisation in the theory of D-branes have been exhaustibly studied in [3]. So we do not elaborate on this. However, our formalism provides an alternate interaction lagrangian picture of the same. The holographic principle [5] is again illustrated—–the finite behaviour of Λ on the boundary gives rise to the exotic solutions within the bulk volume.

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